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NOTE

Remark on Inequalities between Hölder and Lehmer Means

Department of Mathematics and Physics, Anshan Institute of Iron and Steel Technology, Anshan 114002, Liaoning, People's Republic of China

Submitted by William F. Ames

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In [1] Stolarsky established an inequality between the two variable means of Hölder and Lehmer that is much stronger than that given in [2]. However, it should be noted that the inequalities given in [2] are in connection with more general n-variable means. So, it is natural to ask if we could generalize the result of Stolarsky to the n-variable means of Hölder and Lehmer with $n \geq 3$. Unfortunately, the answer is negative.

Let $\boldsymbol{R}$, $\boldsymbol{R}_+$, and $\boldsymbol{N}$ be the set of real numbers, positive numbers, and natural numbers, respectively.

For $n \in \boldsymbol{N}$, $x_1, x_2, \dots, x_n \in \boldsymbol{R}_+$, and $t \in \boldsymbol{R}$, the n-variable Hölder mean of order t is defined by

$$H_t(x_1, x_2, \dots, x_n) = \begin{cases} \left(\dfrac{x_1^t + x_2^t + \cdots + x_n^t}{n} \right)^{\frac{1}{t}}, & t \neq 0, \\ (x_1, x_2 \cdots x_n)^{\frac{1}{n}}, & t = 0, \end{cases}$$

and the n-variable Lehmer mean of order t is defined by

$$L_t(x_1, x_2, \dots, x_n) = \frac{x_1^{t+1} + x_2^{t+1} + \cdots + x_n^{t+1}}{x_1^t + x_2^t + \cdots + x_n^t}.$$

It is well known that for each $x_1, x_2, \dots, x_n \in \boldsymbol{R}_+$, both $H_t(x_1, x_2, \dots, x_n)$ and $L_t(x_1, x_2, \dots, x_n)$ are continuous nondecreasing functions of t for $-\infty < t < +\infty$ and are strictly increasing unless all the x_i are equal [2, 3].

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We now extend [1, Theorem 1] to more comprehensive cases as follows:

THEOREM 1. *Let $x_1, x_2 \in \boldsymbol{R}_+$ and $t \in \boldsymbol{R}$. If $t \in (-1, -\frac{1}{2}) \cup (0, +\infty)$, then*

$$H_{2t+1}(x_1, x_2) \leq L_t(x_1, x_2). \tag{1}$$

If $t \in (-\infty, -1) \cup (-\frac{1}{2}, 0)$, then

$$H_{2t+1}(x_1, x_2) \geq L_t(x_1, x_2). \tag{2}$$

Equality holds when $t = -1, -\frac{1}{2}$, or 0; otherwise equality can hold only when $x_1 = x_2$. Moreover, (1) *is best possible in the sense that $2t + 1$ cannot be replaced by any larger number when $t \in (-1, -\frac{1}{2}) \cup (0, +\infty)$, and* (2) *is best possible in the sense that $2t + 1$ cannot be replaced by any smaller number when $t \in (-\infty, -1) \cup (-\frac{1}{2}, 0)$.*

The proof is almost the same as in proving [1, Theorem 1]. We here only need to note that $H_M(1 + \varepsilon, 1) \geq L_N(1 + \varepsilon, 1)$ for ε small implies $M - 1 \geq N$ and

$$t(2t + 1)x_1^{2t+1} - (t + 1)(2t + 1)x_1^{2t} + (t + 1) \geq 0$$

if $t \in (-1, -\frac{1}{2}) \cup (0, +\infty)$ as well as

$$t(2t + 1)x_1^{2t+1} - (t + 1)(2t + 1)x_1^{2t} + (t + 1) \leq 0$$

if $t \in (-\infty, -1) \cup (-\frac{1}{2}, 0)$ for $x_1 \geq 1$.

Also, it is worth noticing that we can use [4, Theorem 3] to prove this theorem in a different way.

Clearly, inequality (1) is much stronger than that given in [2] for $t > 0$ and inequality (2) is much stronger than that given in [2] for $t < 0$.

THEOREM 2. *Let $x_1, x_2, \ldots, x_n \in \boldsymbol{R}_+$, $n \in \boldsymbol{N}$, and $t \in \boldsymbol{R}$. Then*

$$H_{2t+1}(x_1, x_2, \ldots, x_n) \quad \text{and} \quad L_t(x_1, x_2, \ldots, x_n)$$

are not comparable for $n \geq 3$ and $t \neq -1, 0$.

Proof. First, let $x_1 = \cdots = x_{n-1} = 1$, $x_n = 1 + \varepsilon$, and $\varepsilon > 0$ sufficiently small. Then

$$\begin{aligned} H_{2t+1}(x_1, x_2, \ldots, x_n) &= 1 + \frac{1}{n}\varepsilon + \frac{(n-1)t}{n^2}\varepsilon^2 \\ &\quad + \frac{(n-1)t[2(n-2)t - (n+1)]}{3n^3}\varepsilon^3 + O(\varepsilon^4), \end{aligned}$$

$$\begin{aligned} L_t(x_1, x_2, \ldots, x_n) &= 1 + \frac{1}{n}\varepsilon + \frac{(n-1)t}{n^2}\varepsilon^2 \\ &\quad + \frac{(n-1)t[(n-2)t - n]}{2n^3}\varepsilon^3 + O(\varepsilon^4), \end{aligned}$$

and we see that

$$H_{2t+1}(x_1, x_2, \ldots, x_n) - L_t(x_1, x_2, \ldots, x_n)$$
$$= \frac{1}{6n^3}(n-1)(n-2)t(t+1)\varepsilon^3 + O(\varepsilon^4).$$

Hence, for $n \geq 3$ and $t \neq -1, 0$,

$$H_{2t+1}(x_1, x_2, \ldots, x_n) < L_t(x_1, x_2, \ldots, x_n)$$

if $t \in (-1, 0)$ and

$$H_{2t+1}(x_1, x_2, \ldots, x_n) > L_t(x_1, x_2, \ldots, x_n)$$

if $t \in (-\infty, -1) \cup (0, +\infty)$.

Second, let $x_1 = \cdots = x_{n-1} = 1$, $x_n = 1 - \varepsilon$, and $\varepsilon > 0$ sufficiently small. Then

$$H_{2t+1}(x_1, x_2, \ldots, x_n) = 1 - \frac{1}{n}\varepsilon + \frac{(n-1)t}{n^2}\varepsilon^2 - \frac{(n-1)t[2(n-2)t - (n+1)]}{2n^3}\varepsilon^3 + O(\varepsilon^4),$$

$$L_t(x_1, x_2, \ldots, x_n) = 1 - \frac{1}{n}\varepsilon + \frac{(n-1)t}{n^2}\varepsilon^2 - \frac{(n-1)t[(n-2)t - n]}{2n^3}\varepsilon^3 + O(\varepsilon^4),$$

and we see that

$$L_t(x_1, x_2, \ldots, x_n) - H_{2t+1}(x_1, x_2, \ldots, x_n)$$
$$= \frac{1}{6n^3}(n-1)(n-2)t(t+1)\varepsilon^3 + O(\varepsilon^4).$$

Hence, for $n \geq 3$ and $t \neq -1, 0$,

$$H_{2t+1}(x_1, x_2, \ldots, x_n) > L_t(x_1, x_2, \ldots, x_n)$$

if $t \in (-1, 0)$ and

$$H_{2t+1}(x_1, x_2, \ldots, x_n) < L_t(x_1, x_2, \ldots, x_n)$$

if $t \in (-\infty, -1) \cup (0, +\infty)$.

Thus we can conclude that for $n \geq 3$ and $t \neq -1, 0$, $H_{2t+1}(x_1, x_2, \ldots, x_n)$ and $L_t(x_1, x_2, \ldots, x_n)$ are not comparable.

Finally, we would like to give another interesting result:

THEOREM 3. *Let* $x_1, x_2, \ldots, x_n \in \boldsymbol{R}_+$, $n \in \boldsymbol{N}$, *and* $t \in \boldsymbol{R}$. *Then*

$$\frac{L_t(x_1, x_2, \ldots, x_n) + L_{-(t+1)}(x_1, x_2, \ldots, x_n)}{2} \geq H_0(x_1, x_2, \ldots, x_n)$$

holds for $n = 2$ *and may not hold for* $n \geq 3$.

Proof. For $n = 2$, it is immediate that

$$\frac{L_t(x_1, x_2) + L_{-(t+1)}(x_1, x_2)}{2} = \frac{1}{2}\left(\frac{x_1^{t+1} + x_2^{t+1}}{x_1^t + x_2^t} + \frac{x_1^t + x_2^t}{x_1^{t+1} + x_2^{t+1}} x_1 x_2\right)$$
$$\geq \sqrt{x_1 x_2} = H_0(x_1, x_2).$$

For $n \geq 3$, let $x_1 = \cdots = x_{n-1} = 1$, $x_n = 1 - \varepsilon$, and $\varepsilon > 0$ sufficiently small, then

$$\frac{L_t(x_1, x_2, \ldots, x_n) + L_{-(t+1)}(x_1, x_2, \ldots, x_n)}{2}$$
$$= 1 - \frac{1}{n}\varepsilon - \frac{n-1}{2n^2}\varepsilon^2 - \frac{(n-1)[(n-2)t(t+1) + (n-1)]}{2n^3}\varepsilon^3 + O(\varepsilon^4),$$

$$H_0(x_1, x_2, \ldots, x_n) = 1 - \frac{1}{n}\varepsilon - \frac{n-1}{2n^2}\varepsilon^2 - \frac{(n-1)(2n-1)}{6n^3}\varepsilon^3 + O(\varepsilon^4),$$

and we see that

$$H_0(x_1, x_2, \ldots, x_n) - \frac{L_t(x_1, x_2, \ldots, x_n) + L_{-(t+1)}(x_1, x_2, \ldots, x_n)}{2}$$
$$= \frac{1}{6n^3}(n-1)(n-2)(3t^2 + 3t + 1)\varepsilon^3 + O(\varepsilon^4),$$

and it follows

$$\frac{L_t(x_1, x_2, \ldots, x_n) + L_{-(t+1)}(x_1, x_2, \ldots, x_n)}{2} < H_0(x_1, x_2, \ldots, x_n)$$

for $-\infty < t < +\infty$.

The proof of the theorem is complete.

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